Section 5.4

Saturday, April 4, 2020 9:51 PM

Recall that, a continuous function f on $[a,b]$ is integrable and $\int_a^b f(x)\,dx = I$ if there exists a sequence of partitions $\{P_i\}$ with $P_1 \subseteq P_2 \subseteq P_3 \subseteq \dots$ such that

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n) = I$$

this condition can actually be dropped!

Some properties of the definite integral Let f,g be integrable on a closed interval containing a,b,c.

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b (c_1 f(x) + c_2 g(x))\,dx = c_1 \int_a^b f(x)\,dx + c_2 \int_a^b g(x)\,dx$ for every constant c_1, c_2.
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for every constant c
- If $f(x) \le g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
- $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$

Example: Show that $\frac{3}{4} \le \int_0^1 e^{-x^3}\,dx \le 1$.

Solution: $\boxed{\text{Exercise for the reader: } e^t \ge 1+t \text{ for all } t.}$ We have that

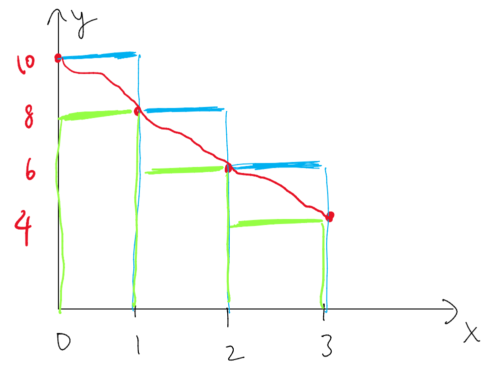
$$1-x^3 \le e^{-x^3} \le 1 \quad \text{for all } x \text{ in } [0,1]. \text{ Thus}$$

$$\int_0^1 (1-x^3)\,dx \le \int_0^1 e^{-x^3}\,dx \le \int_0^1 1\,dx \quad \text{and so,}$$

$$\int_0^1 1\,dx - \underbrace{\int_0^1 x^3\,dx}_{\text{computed last time using Riemann sums}} \le \int_0^1 e^{-x^3}\,dx \le \int_0^1 1\,dx. \text{ This means } 1 - \frac{1}{4} \le \int_0^1 e^{-x^3}\,dx \le 1.$$

Example: Let f be continuous and decreasing on $[0,3]$. Suppose that $f(0)=10$, $f(1)=8$, $f(2)=6$ and $f(3)=4$. Show that $18 \le \int_0^3 f(x) \le 24$.

Solution:



As f is decreasing $[0,3]$, for the partition $P = \{0,1,2,3\}$, we get

$$U(f,P) = 10 \cdot 1 + 8 \cdot 1 + 6 \cdot 1 = 24$$

$$L(f,P) = \underbrace{8}_{\substack{\min f(x) \\ 0 \le x \le 1}} \cdot \underbrace{1}_{\text{the length of } [0,1]} + 6 \cdot 1 + 4 \cdot 1 = 18$$

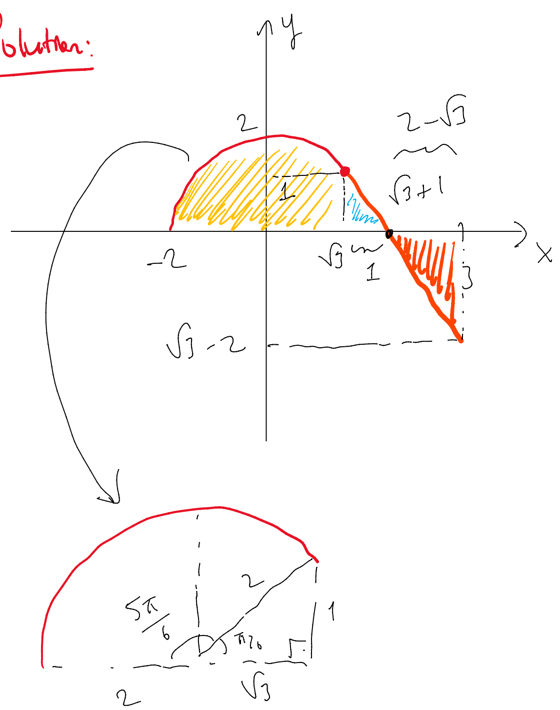
max $f(x)$, $0 \le x \le 1$

By defn, we have

$$18 = L(f,P) \le \int_0^3 f(x)\,dx \le U(f,P) = 24$$

Example: Let $f(x) = \begin{cases} \sqrt{4-x^2} & \text{if } -2 \le x < \sqrt{3} \\ -x+\sqrt{3}+1 & \text{if } \sqrt{3} \le x \le 3 \end{cases}$. Find $\displaystyle\int_{-2}^{3} f(x)\,dx$.

Solution:



We have that

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{\sqrt{3}} f(x)\,dx + \int_{\sqrt{3}}^{\sqrt{3}+1} f(x)\,dx + \int_{\sqrt{3}+1}^{3} f(x)\,dx$$

$$= \left(\pi 2^2 \cdot \frac{\frac{5\pi}{6}}{2\pi} + \frac{1 \cdot \sqrt{3}}{2}\right) + \frac{1 \cdot 1}{2} - \frac{(2-\sqrt{3}) \cdot (2-\sqrt{3})}{2}$$

The mean-value theorem for integrals Let f be continuous on $[a,b]$. Then there exists a number $a \le c \le b$ such that

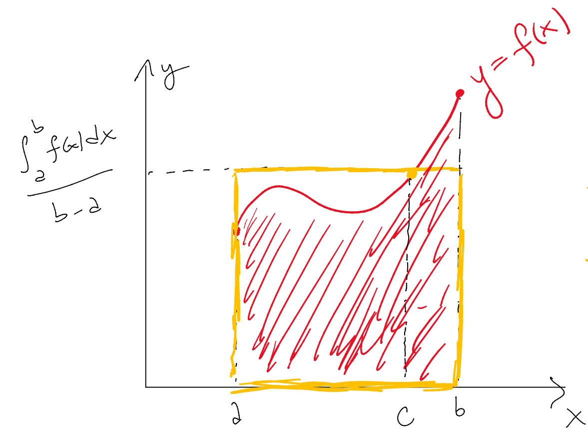
$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

Proof: By the min-max theorem, f attains its absolute max and min values on $[a,b]$, say, M and m respectively. Then, from the partition $P=\{a,b\}$, we get that

$$m\cdot(b-a) = L(f,P) \le \int_a^b f(x)\,dx \le U(f,P) = M\,(b-a) \quad \text{and so}$$

$$m \le \frac{\int_a^b f(x)\,dx}{b-a} \le M.$$

Since f is continuous, by the IVT, there exists $a \le c \le b$ with $f(c) = \dfrac{\int_a^b f(x)\,dx}{b-a}$. $\blacksquare$

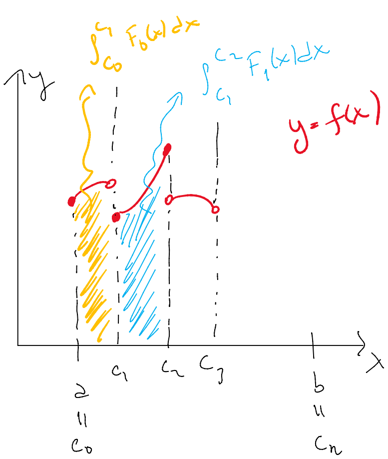


the area of the yellow rectangle $= \int_a^b f(x)\,dx$

The number $\dfrac{\int_a^b f(x)\,dx}{b-a}$ is called *the average (mean) value of f over $[a,b]$*.

The definite integral of piecewise continuous functions

Suppose that f is **piecewise continuous** on $[a,b]$, that is, there are points $a=c_0<c_1<c_2<\cdots<c_n=b$ such that $f(x)=F_i(x)$ on (c_i,c_{i+1}) for some continuous function F_i on $[c_i,c_{i+1}]$. Then we define



$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_a^b F_i(x)\,dx$$